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REDUCING THE EQUATIONS OF MOTION OF CERTAIN NON-HOLONOMIC CHAPLYGIN SYSTEMS TO LAGRANGIAN AND HAMILTONIAN FORM*

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Non-holonomic Chaplygin systems /1/ with n degrees of freedom and m $(m < n)$ first integrals linear with respect to velocities, are considered. It is assumed that Lagrange's function is constructed taking into account the non-holonomic constraints imposed on the system, and the integrals are independent of the first m generalized coordinates. Then, provided that certain conditions are met, m linear non-holonomic coordinates (quasi-coordinates) can be introduced in such a way that the first m equations of motion in these coordinates will have the form of the usual Lagrange's equations.

The present paper deals with the most interesting, integrable case, when $m = n - 1$. It is shown that if certain conditions are met, the trajectories of such a system in phase space will represent quasiperiodic windings on the n-dimensional tori. Examples are given, namely, of a solid of revolution rolling along a fixed horizontal plane, and of the motion of a circular disc with a sharp edge on a smooth, horizontal ice surface.

The problem of reducing Chaplygin's equations of motion of non-holonomic systems to the form of the ordinary Lagrangian and Hamiltonian equations has been studied extensively. A detailed survey and an analysis of the existing approaches to solving this problem are given in /2/.

1. Let us consider a natural, non-holonomic mechanical Chaplygin system /1/ acted upon by potential forces. We assume that Lagrange's function constructed taking into account the non-integrable constraints imposed on the system, has the form

$$L(\mathbf{q}, \mathbf{q}^{\cdot}) = T - \Pi, \quad T = {}^1/_2 \mathbf{q}^{\cdot T} \boldsymbol{\Omega} \mathbf{q}^{\cdot}, \quad \Pi = \Pi(\mathbf{q}) \qquad (1.1)$$
$$\boldsymbol{\Omega} = \| \omega_{ij}(\mathbf{q}) \| \quad (i, j = 1, 2, \ldots, n)$$

Here $\mathbf{q}$, $\mathbf{q}^{\cdot}$ are column matrices of the generalized coordinates and velocities of the system, $\boldsymbol{\Omega}$ is a positive definite symmetric $n \times n$-matrix, T and Π is the kinetic and potential energy of the system respectively. The total energy of the system is conserved $(T + \Pi = h = \text{const})$, and the differential Chaplygin equations of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \mathbf{q}^{\cdot}} - \frac{\partial L}{\partial \mathbf{q}} = \boldsymbol{\Gamma} \qquad (1.2)$$

will describe the motion of the system independently of the equations of non-integrable constraints. In (1.2) $\boldsymbol{\Gamma}$ is a column matrix of the non-holonomic terms ($\Gamma_i(\mathbf{q}, \mathbf{q}^{\cdot})$ is the quadratic

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form of the velocities $q^{\cdot}_j$).

We can also write the Chaplygin equations in canonical form

$$\mathbf{p}^{\cdot} = -\partial H/\partial \mathbf{q} + \mathbf{\Phi}, \ \mathbf{q}^{\cdot} = \partial H/\partial \mathbf{p} \tag{1.3}$$

where $\mathbf{p} = \partial L/\partial \mathbf{q}^{\cdot}$ and the functions $H(\mathbf{p}, \mathbf{q})$ and $\Phi_i(\mathbf{p}, \mathbf{q})$ are obtained from the functions $L(\mathbf{q}, \mathbf{q}^{\cdot})$ and $\Gamma_i(\mathbf{q}, \mathbf{q}^{\cdot})$ by replacing the generalized velocities $\mathbf{q}^{\cdot}$ by generalized moments $\mathbf{p}$.

Let $\partial L/\partial q_k = 0$ $(k = 1, \ldots, m < n)$. Unlike in the holonomic systems, this does not lead to the first integrals of the equations of motion linear with respect to the velocities. In a number of problems however, the integrals can be determined /2, 3/.

We further assume that there exist exactly m independent first integrals linear with respect to the velocities

$$\mathbf{I} = \mathbf{\Lambda q}^{\cdot}, \quad \mathbf{\Lambda} = \| \lambda_{kj}(\mathbf{q}) \| \tag{1.4}$$

Here $\mathbf{I}$ is the column matrix of the first integrals (we assume for convenience that it has the dimension of the generalized moments), and we assume that $\partial\lambda_{kj}/\partial q_l = 0$ $(l = 1, \ldots, m)$.

In what follows, we shall make use of the following representations of the matrices $\mathbf{\Lambda}$ and $\mathbf{\Omega}$:

$$\mathbf{\Omega} = \begin{Vmatrix} \mathbf{\Omega}_1 & \mathbf{\Omega}_2 \\ \mathbf{\Omega}_2^T & \mathbf{\Omega}_3 \end{Vmatrix}, \quad \mathbf{\Lambda} = \| \mathbf{\Lambda}_1, \mathbf{\Lambda}_2 \|, \quad \mathbf{\Omega}_1 = \| \omega_{kl} \|, \quad \mathbf{\Lambda}_1 = \| \lambda_{kl} \| \tag{1.5}$$

and we assume that at the points of general position $\det \mathbf{\Lambda}_1 \neq 0$.

Let us choose the quasicoordinates π_k so that the corresponding moments are the first integrals of the equations of motion of the Chaplygin system in question. To do this, we will carry out the following substitution:

$$\mathbf{q}^{\cdot} = \mathrm{S}\boldsymbol{\pi}^{\cdot}, \quad \mathrm{S} = \begin{Vmatrix} \mathrm{S}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{Vmatrix} \tag{1.6}$$

Here $\mathrm{S}_1(\mathbf{q})$ is a non-degenerate $m \times m$-matrix, and $\mathbf{E}$ is the unit matrix. From (1.6) we see that the last $n - m$ quasicoordinates are identical with the initial generalized coordinates (we only change the notation for convenience), and in place of the first m coordinates q_k we will consider the linear quasicoordinates.

Lagrange's function will have the following form in quasicoordinates:

$$L^* = T^* - \Pi^*, \quad T^* = {}^1/_2 \boldsymbol{\pi}^{\cdot T} \mathbf{\Psi} \boldsymbol{\pi}^{\cdot}, \quad \mathbf{\Psi} = (\mathrm{S}^T \mathbf{\Omega} \mathrm{S})^* \tag{1.7}$$

and the equation of motion will be

$$\frac{d}{dt}\frac{\partial L^*}{\partial \boldsymbol{\pi}^{\cdot}} - \frac{\partial L^*}{\partial \boldsymbol{\pi}} = \mathbf{\Gamma}^*(\boldsymbol{\pi}, \boldsymbol{\pi}^{\cdot}) \tag{1.8}$$

We shall denote by an asterisk the passage from the coordinates $\mathbf{q}$ to the quasicoordinates $\boldsymbol{\pi}$ (or conversely, from the quasicoordinates $\boldsymbol{\pi}$ to the coordinates $\mathbf{q}$). We note that

$$\frac{\partial L^*}{\partial \pi_k} = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} s_{ik} \right)^* = 0 \tag{1.9}$$

Thus the function $L^*(\boldsymbol{\pi}, \boldsymbol{\pi}^{\cdot})$ does not depend explicitly on the first m quasicoordinates (and the last $n - m$ coordinates are identical with the initial coordinates), i.e., $\Pi^* = \Pi^*(\pi_{m+1}, \ldots, \pi_n)$, $S^* = S^*(\pi_{m+1}, \ldots, \pi_n)$ in (1.7).

Now let $\mathrm{S}_1 = \mathbf{\Omega}_1^{-1}\mathbf{\Lambda}_1^T$. Then, provided that the relation

$$\mathbf{\Lambda}_1^{-1}\mathbf{\Lambda}_2 = \mathbf{\Omega}_1^{-1}\mathbf{\Omega}_2 \tag{1.10}$$

holds, we have in the first m equations of (1.8) $\Gamma_i^* = 0$. Indeed,

$$\left(\frac{\partial L^*}{\partial \boldsymbol{\pi}^{\cdot}}\right)^* = [(\mathrm{S}^T \mathbf{\Omega} \mathrm{S})^* \boldsymbol{\pi}^{\cdot}]^* = \mathrm{S}^T \mathbf{\Omega} \mathrm{S} (\mathrm{S}^{-1}\mathbf{q}^{\cdot}) = \mathrm{S}^T \mathbf{\Omega} \mathbf{q}^{\cdot} = \begin{Vmatrix} \mathrm{S}_1^T \mathbf{\Omega}_1 & \mathrm{S}_1^T \mathbf{\Omega}_2 \\ \mathbf{E}\mathbf{\Omega}_2^T & \mathbf{\Omega}_3 \end{Vmatrix} \mathbf{q}^{\cdot} = \begin{Vmatrix} \mathbf{\Lambda}_1 & \mathbf{\Lambda}_2 \\ \mathbf{E}\mathbf{\Omega}_2^T & \mathbf{\Omega}_3 \end{Vmatrix} \mathbf{q}^{\cdot} \tag{1.11}$$

From (1.11) it follows that $\partial L^*/\partial \pi_k^{\cdot} = I_k{}^*{}_{\mathfrak{t}} = \mathrm{const}$. Therefore $(d/dt)(\partial L^*/\partial \pi_k^{\cdot}) = 0$ and taking into account (1.9) we conclude that $\Gamma_k^* = 0$ in (1.8). Consequently, we have the following theorem.

Theorem 1. If a non-holonomic Chaplygin system with n degrees of freedom has m $(m < n)$ independent first integrals linear with respect to velocities, such that neither these integrals, nor Lagrange's function constructed taking into account the non-holonomic constraints, depend on the first m coordinates, then, provided that some relation (1.10) holds, the first

m coordinates can be replaced by the quasicoordinates in such a manner that the non-holonomic terms in the first m equations vanish.

Notes. 1°. Condition (1.10) is automatically satisfied if $\Lambda_2 = 0$, $\Omega_2 = 0$, i.e. when the first integrals depend only on the first m velocities q_k and the expression for the kinetic contains no terms of the form $\omega_{k\nu} q_k q_\nu$ ($\nu = m+1, \ldots, n$). The condition is invariant under the point transformation of the coordinates $q_{m+1}, \ldots, q_n$.

2°. If the matrix S_1 contains integrable rows, then the corresponding quasicoordinate is simply a generalized coordinate.

3°. The equations of motion (1.8) can also be written in canonical form /4/

$$\mathbf{P}^{\cdot} = -\partial H^*/\partial \boldsymbol{\pi} + \boldsymbol{\Phi}^*, \quad \boldsymbol{\pi}^{\cdot} = \partial H^*/\partial \mathbf{P} \tag{1.12}$$

where $\mathbf{P} = \partial L^*/\partial \boldsymbol{\pi}^{\cdot}$ and $H^*(\mathbf{P}, \boldsymbol{\pi})$, $\Phi_i^*(\mathbf{P}, \boldsymbol{\pi})$ are obtained from the functions $L^*(\boldsymbol{\pi}, \boldsymbol{\pi}^{\cdot})$ and $\Gamma_i^*(\boldsymbol{\pi}, \boldsymbol{\pi}^{\cdot})$ as a result of replacing $\boldsymbol{\pi}^{\cdot}$ by $\mathbf{P}$, with $\Phi_k^* = 0$.

4°. The q_i may already include the quasicoordinates. It is only important that Lagrange's function (1.1) should not depend explicitly on them. If all conditions of Theorem 1 hold here, then there are no changes in the arguments used.

5°. The problem of using the quasicoordinates in non-holonomic mechanics has not been sufficiently explored /2/. We therefore must introduce the quasicoordinates with extreme care. Generally speaking, the relation between the quasicoordinates and the initial (true) coordinates can only be established for a specific trajectory of motion (which is, generally speaking, not known). In the problem discussed here we manage to establish this relationship for the case $m = n - 1$.

2. Let us consider the case $m = n - 1$ in more detail. The study of this integrable case leads to an analysis of a one-dimensional system (with a single local coordinate $\pi_n = q_n$) whose energy is conserved. Theorem 1 implies at once that in (1.12) the only function which can be different from zero, is Φ_n^*. We shall, however, show that we also have $\Phi_n^* = 0$, i.e. $P_n^{\cdot} = -\partial H^*/\partial \pi_n$.

The Hamiltonian function in the quasicoordinates has the form

$$H^* = {}^1/_2 \mathbf{P}^T \Psi^{-1}(\pi_n) \mathbf{P} + \Pi(\pi_n) \tag{2.1}$$

Differentiating (2.1) in time we obtain

$$\frac{\partial H^*}{\partial \pi_n} \pi_n^{\cdot} + \frac{\partial H^*}{\partial P_n} P_n^{\cdot} = 0 \tag{2.2}$$

and from this we have

$$\pi_n^{\cdot} \left(P_n^{\cdot} + \frac{\partial H^*}{\partial \pi_n} \right) = 0 \tag{2.3}$$

When $\pi_n^{\cdot} \not\equiv 0$, (2.3) yields the required relation at once. If, on the other hand, $\pi_n^{\cdot} \equiv 0$, then $P_n^{\cdot} = -\partial H^*/\partial \pi_n = 0$. Therefore in this case we have $\boldsymbol{\Phi}^* = 0$, in (1.12), i.e. the equations of motion have the form of the ordinary Hamiltonian equations

$$\mathbf{P}^{\cdot} = -\partial H^*/\partial \boldsymbol{\pi}, \; \boldsymbol{\pi}^{\cdot} = \partial H^*/\partial \mathbf{P} \tag{2.4}$$

Thus we reduce the investigation of the non-holonomic Chaplygin system in question to the study of a Hamiltonian system with n degrees of freedom and $n-1$ ignorable coordinates. The motion of such a system has already been studied in detail (see e.g. /5/).

We shall show that a smooth reversible change of the variables $\mathbf{q} = \mathbf{q}(\boldsymbol{\pi}, \mathbf{P})$, $\mathbf{p} = \mathbf{p}(\boldsymbol{\pi}, \mathbf{P})$ exists and that it can be used to reduce the equations of motion of the holonomic system (1.3) to the form (1.4).

First we note that (2.1) will always yield, with help of the energy integral, $\pi_n^{\cdot}$ in terms of $h, P_1, \ldots, P_{n-1}$ and π_n:

$$\pi_n^{\cdot} = U(h, P_1, \ldots P_{n-1}, \pi_n)$$

From (1.1), (1.6) and (1.7) it follows that

$$\mathbf{p} = \boldsymbol{\Omega}\mathbf{q}^{\cdot} = (\boldsymbol{\Omega}\mathbf{S})^* \boldsymbol{\pi}^{\cdot} = [\mathbf{S}^T(\pi_n)]^{-1} \mathbf{P} \tag{2.5}$$

Let us now find the relation connecting $\mathbf{q}$ with $\boldsymbol{\pi}$ and $\mathbf{P}$. We introduce the function

$$\mathbf{F}(h, P_1, \ldots, P_{n-1}, \pi_n) = \int \frac{\mathbf{W}\mathbf{P}}{U} d\pi_n \tag{2.6}$$

$$\mathbf{W}(\pi_n) = [(\mathbf{E} - \mathbf{S}^{-1}) \boldsymbol{\Omega}^{-1} (\mathbf{S}^T)^{-1}]^*$$

We note that the last row and column in the matrix $\mathbf{W}$ are both zero, i.e. $F_n = 0$ and $\mathbf{W}\mathbf{P}$ is independent of P_n.

The change of variables sought will have the form

$$\mathbf{q} = \boldsymbol{\pi} + \mathbf{Q}(\mathbf{P}, \pi_n) \tag{2.7}$$

where the function $\mathbf{Q}$ is found from $\mathbf{F}$ by replacing the total energy constant h by $H^*(\mathbf{P}, \pi_n)$, i.e. $\mathbf{Q}(\mathbf{P}, \pi_n) \equiv \mathbf{F}[H^*(\mathbf{P}, \pi_n), P_1, \ldots, P_{n-1}, \pi_n]$.

Let us confirm this. Indeed, differentiating (2.7) with respect to time we obtain, by virtue of the equations of motion,

$$\mathbf{q}^{\cdot} = \boldsymbol{\pi}^{\cdot} + \mathbf{Q}^{\cdot}(\mathbf{P}, \pi_n) = \boldsymbol{\pi}^{\cdot} + \mathbf{W}\mathbf{P}\pi_n^{\cdot}/U = \boldsymbol{\pi}^{\cdot} + \mathbf{W}\mathbf{P} = \boldsymbol{\pi}^{\cdot} + \\ [(\mathbf{E} - \mathbf{S}^{-1})\,\boldsymbol{\Omega}^{-1}(\mathbf{S}^T)^{-1}(\mathbf{S}^T\boldsymbol{\Omega}\mathbf{S})]^*\,\boldsymbol{\pi}^{\cdot} = \boldsymbol{\pi}^{\cdot} + (\mathbf{S}^* - \mathbf{E})\,\boldsymbol{\pi}^{\cdot} = \mathbf{S}^*\boldsymbol{\pi}^{\cdot} \tag{2.8}$$

From (2.5)-(2.7) it follows that the change $(\mathbf{p}, \mathbf{q}) \to (\mathbf{P}, \boldsymbol{\pi})$ is fairly smooth and reversible, and its Jacobian (equal to $\det \mathbf{S}$) is different from zero.

Thus we have, at every fixed level of the first integrals, a single-valued relation connecting $\mathbf{q}$ and $\boldsymbol{\pi}$, i.e. the quasicoordinates $\boldsymbol{\pi}$ determine completely the position of the system at every level of the first integrals.

Let us now consider the structure of the phase space of the non-holonomic Chaplygin system for this case.

If M_0 is a configurational space of the initial system and L_0 is its Lagrange's function, i.e. L_0: $TM_0 \to R$ (TM_0 is the tangential stratification of M_0), then since the system is Chaplygin-type, there exists $M \subset M_0$ (we assume that M is a manifold) such that the mapping $\delta: R \to M$, can be regarded as the motion of the system in question. The mapping must satisfy, in the local coordinates $\mathbf{q}$ on M, either the Chaplygin Eqs.(1.2) with the Lagrangian $L: TM \to R$ (L is Lagrange's function constructed taking into account the non-integrable constraints imposed on the system), or the Chaplygin Eqs.(1.3) in canonical form with the function $H: T^*M \to R$ (T^*M is the cotangential stratification of M). We find that the coordinates $\mathbf{P}, \boldsymbol{\pi}$ can be introduced in the phase space T^*M of the non-holonomic Chaplygin system in question (with the local coordinates $\mathbf{p}, \mathbf{q}$) in such a manner, that the equations of motion in these coordinates will have the form of the ordinary Hamiltonian equations. The Hamiltonian system obtained in this manner has n independent first integrals in the involution, and is completely Liouville-integrable /5/.

If the non-singular set of the level of first integrals is compact and connected, then it is diffeomorphic to the n-dimensional torus T^n and the trajectories of motion are quasi-periodic windings on this torus. As in every Liouville-integrable Hamiltonian system, we can further introduce the action-angle variables. We note that the angle coordinates $\mathbf{w}$ on the torus and coordinates $\mathbf{q}$ will be connected by a relation analogous to (2.7), i.e. $\mathbf{w} = \mathbf{q} + \mathbf{Q}'(h, P_1, \ldots, P_{n-1}, q_n)$.

If we have $\partial U/\partial \pi_n = 0$ at some level of the first integrals, then this level will be "singular" and the independence of the first integrals will be violated on it. In this case the non-holonomic system in question may contain a stationary motion $q_n = c_n$, $q_k^{\cdot} = c_k$ (c_i are certain constants) which cannot be asymptotically stable with respect to some of the variables /6/.

Therefore the following theorem holds.

Theorem 2. Let a natural non-holonomic Chaplygin system with n degrees of freedom have $n-1$ first integrals linear with respect to velocities. The integrals and the Lagrange's function constructed taking into account the non-holonomic constraints, depend on a single coordinate q_n. Moreover, let condition (1.10) hold. We consider the set of the level of first integrals in Σ. If all n integrals are independent of Σ and Σ is compact and connected, then $\Sigma \simeq T^n$ and the trajectories of motion will be quasiperiodic windings on this torus. There exists a smooth change of the "real" canonical variables $(\mathbf{p}, \mathbf{q})$ reducing the equations of motion of the non-holonomic Chaplygin system in question to the ordinary Hamiltonian equations.

Corollary. The non-holonomic Chaplygin systems satisfying the conditions of Theorem 2, have an integral invariant whose density is $\mu = \mu(q_n)$.

Note. If the non-holonomic Chaplygin system with two degrees of freedom admits, in addition to the energy integral, of another first integral, and the set Σ of the non-singular level of thefirst integrals is compact and connected, then it follows at once that $\Sigma \simeq T^2$ /7/ (since Σ is a compact, orientable two-dimensional manifold admitting a vector field without singularities).

3. Let us consider a problem of a heavy convex solid of revolution rolling along a horizontal plane in a homogeneous gravity field /1/. The configurational space of the system $M_0 = R^2 \times SO(3)$. We choose the local coordinates on M_0 as follows: ξ, η are the coordinates of the projection of the centre of gravity G of the body onto the horizontal plane in the

fixed coordinate system $O\xi\eta\zeta$ (the $O\zeta$ axis is directed vertically upwards) and ψ, θ, φ are the Euler angles characterizing the orientation of the coordinate system $Gxyz$ rigidly attached to the body (we direct the x, y, z axes along the principal central axes of inertia of the body, and the z axis along axis of symmetry) relative to the fixed system. Two non-integrable constraints are imposed on the system and the absolute velocity of the point of the body coinciding with the point of contact, is zero.

In this case $M = SO(3)$ and the Lagrangian function L constructed taking into account the non-holonomic constraints, has the form (1.1) where

$$\mathbf{q}^T = (\psi, \varphi, \theta),\quad \Pi(\theta) = mgf(\theta),\quad \Omega_2^T = (0, 0),\quad \Omega_3 = A + m(f^2 + \rho^2) \tag{3.1}$$
$$\Omega_1 = \begin{Vmatrix} A\sin^2\theta + C\cos^2\theta + m\rho^2 & C\cos\theta + m\rho\chi \\ C\cos\theta + m\rho\chi & C + m\chi^2 \end{Vmatrix},\quad \chi = f\sin\theta + \rho\cos\theta$$

Here A, A, C are the moments of inertia of the body about the x, y, z axes, m is the mass of the body, g is the acceleration due to gravity, $f(\theta)$ is the height of the centre of gravity above the plane and $\rho = df/d\theta$.

We know /1, 8/ that the equations of motion (1.2) admit in this case of two integrals, $\psi^\cdot$ and $\varphi^\cdot$, linear in velocities and depending explicitly only on θ. Thus we have in (1.4) $\Lambda_1 = \Lambda_1(\theta)$, and $\Lambda_2 = 0$. Therefore all the conditions of Theorem 2 hold and the quasiperiodic windings on the three-dimensional torus are the trajectories of motion in the phase space $T^*SO(3)$.

In the case of an arbitrary function $f(\theta)$ the explicit form of these integrals (i.e. the matrices Λ_1) is not known, and therefore we shall carry out our investigation for the case when the body is bounded by a sphere of radius $d > 0$. Then /1/ $f(\theta) = d + r\cos\theta$ ($|r| < d$) and

$$\Lambda_1 = A\begin{Vmatrix} \beta\cos\theta + \sin^2\theta & \beta \\ \alpha\cos\theta & \alpha \end{Vmatrix},\quad \beta(\theta) = \gamma\left(\frac{r}{d} + \cos\theta\right) \tag{3.2}$$
$$\alpha(\theta) = \left(\frac{C}{md^2} + \sin^2\theta + \frac{\beta^2}{\gamma}\right)^{1/2},\quad \gamma = \frac{C}{A}$$

From (3.2) we find that $\det\Lambda_1 = A\alpha(\theta)\sin^2\theta$. Since $\alpha > 0$, it follows that $\det\Lambda_1 = 0$ when $\sin\theta = 0$. It can, however, be shown that this is related only to the particular character of the coordinate system used.

We obtain the following expression for the matrix S_1:

$$S_1 = \begin{Vmatrix} 1 & \beta/(\alpha\gamma) \\ r/d & (C + mdr\beta)/(md^2\alpha\gamma) \end{Vmatrix},\quad \det S_1 = \frac{A}{\alpha md^2} > 0 \tag{3.3}$$

Thus the problem reduces to the study of a Hamiltonian system with the Hamiltonian function

$$H^*(\pi_1, \pi_2, \theta, P_1, P_2, P_3) = \frac{1}{2A}\left[\frac{md^2}{C}\left(\frac{\alpha P_1 - \beta P_2}{\sin\theta}\right)^2 + md^2\left(\frac{P_2^2}{A} - \frac{P_1^2}{C}\right) + \right. \tag{3.4}$$
$$\left.\left[1 + \frac{md^2}{A}\left(1 + 2\frac{r}{d}\cos\theta + \frac{r^2}{d^2}\right)\right]^{-1} P_3^2\right] + mgr\cos\theta$$

Further investigation can be carried out just as in the case when a solid of revolution moves along a perfectly smooth surface.

We note that this axis of symmetry of the body Gz can pass through the vertical position, provided that the relation /1/ $\alpha(0)P_1 - \beta(0)P_2 = 0$ or $\alpha(\pi)P_1 - \beta(\pi)P_2 = 0$ holds.

Let us point out a certain analogy between the problem of rolling a homogeneous sphere and the problem of geodesics on a sphere. Let $A = C = (1 + \sqrt{5})md^2/2$, $r = 0$, $P_1 = 0$ in (3.4) (the projection of the vector of angular momentum onto the vertical is equal to zero), then the Hamiltonian (3.4) will have exactly the same form as the Hamiltonian of the problem of motion of a material point of mass $m(3 + \sqrt{5})/2$ over a smooth retaining sphere of radius d.

If we write the equations of motion of the non-holonomic Chaplygin system in question in the form (1.3) using the real canonical variables $\mathbf{q}^T = (\psi, \varphi, \theta)$, $\mathbf{P}^T = (p_\psi, p_\varphi, p_\theta)$, they will admit of the last Jacobi multiplier $\mu(\theta) = \alpha^{-1}(\theta)$ and the divergence of the vector field specified by the right-hand side of these equations will be euqla to $\varkappa = p_\theta(\cos\theta - \beta)\sin\theta/(\alpha^2\omega_{33})$. Thus both the density of the integral invariant μ and $\varkappa$ are periodic along any trajectory lying on a non-singular level of the first integrals H, I_1, I_2.

4. As the second example, we shall consider the problem of a circular disc with a sharp edge moving on smooth horizontal ice /9/. The disc moves without cutting the ice, i.e. the velocity of the point of the disc which coincides with the point of contact is parallel to its horizontal diameter.

Retaining the notation used in the previous example, we have $f(\theta) = d\sin\theta$ (d is the radius of the disc). We replace the generalized coordinates ξ, η by the quasilinear coordinates σ_1, σ_2

$$\sigma_1^\cdot = \xi^\cdot\cos\psi + \eta^\cdot\sin\psi,\quad \sigma_2^\cdot = -\xi^\cdot\sin\psi + \eta^\cdot\cos\psi \tag{4.1}$$

The following constraint is imposed on the system:

$$\sigma_2 = d\cos\theta \tag{4.2}$$

The non-holonomic Chaplygin system in question has four degrees of freedom. Here $M = R^1 \times SO(3)$ and the Lagrange's function L constructed taking into account the non-holonomic constraints, has the form (1.1) where

$$\mathbf{q}^T = (\psi, \varphi, \sigma_1, \theta),\ \Pi(\theta) = mgd\sin\theta,\ \mathbf{\Omega}_2 = \mathbf{0},\ \mathbf{\Omega}_3 = A + md^2 \tag{4.3}$$

$$\mathbf{\Omega}_1 = \begin{Vmatrix} A\sin^2\theta + C\cos^2\theta & C\cos\theta & 0 \\ C\cos\theta & C & 0 \\ 0 & 0 & m \end{Vmatrix}$$

We know /9/ that the problem has three integrals linear in $\psi^{\cdot}, \varphi^{\cdot}, \sigma_1^{\cdot}$, i.e. $\Lambda_2 = \mathbf{0}$, in (1.4) and

$$\lambda_{11} = \lambda_{22} = \omega_{12},\ \lambda_{12} = \omega_{22},\ \lambda_{13} = \lambda_{23} = 0,\ \lambda_{21} = \omega_{11} \tag{4.4}$$

$$\lambda_{31} = md^2\,[(1 + \gamma\ln\sin\theta)\cos\theta - (\sin^2\theta + \gamma\cos^2\theta)\ln\operatorname{tg}{}^1/_2\theta]$$

$$\lambda_{32} = md^2\,[1 + \gamma(\ln\sin\theta - \cos\theta\ln\operatorname{tg}{}^1/_2\theta)],\ \ \lambda_{33} = md$$

In this case the set Σ is not compact. All the remaining conditions of Theorem 2, however, hold. Therefore, there exists a change of variables which reduces the equations of motion of the system in question to the form of the ordinary Hamiltonian equations, and the trajectories of motion in the phase space T^*M represent the windings of a four-dimensional cylinder $R^1 \times T^3$ /5/.

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